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Heat-Flux Comparison Based on Properties of the Medium

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1. INTRODUCTION

The purpose of this paper is to present a certain comparison result for solutions $u^{(i)}(x,t)$, $i = 1,2$, of equations

$$u_t^{(i)} = \left(k^{(i)}(u^{(i)})u_x^{(i)}\right)_x, \tag{1.1}$$

in a domain $G = \{(x,t) \in \mathbb{R}^2 : 0 \le x_1 < x < x_2, 0 < t \le T\}$, with the proviso that $u^{(i)}(x,t)$ be monotone (decreasing) in the space variable x for fixed t, and $k^{(i)}$ satisfy condition (1.2), below.

If the level curves $u^{(1)}(x,t) = w$, $u^{(2)}(x,t) = w$ are defined for all t and contained in G, the main result (Theorem 1 and inequality (2.11), Section 2) reads

$$\int_0^t \left(-k^{(1)}(w)u_x^{(1)}\Big|_{u^{(1)}=w}\right) ds \ge \int_0^t \left(-k^{(2)}(w)u_x^{(2)}\Big|_{u^{(2)}=w}\right) ds,$$

if $k^{(1)} \ge k^{(2)}$ for all values of their argument. That is, an inequality for the total flux of heat across the level curves at temperature w, up to time t, is a consequence of the corresponding inequality satisfied by the diffusivities $k^{(i)}$.

This inequality is employed to estimate the position of the moving boundary of one problem of the Stefan type in terms of that of another (cf. Section 3, Theorem 2); some cases in which the latent heats of the change of phase $\lambda^{(1)}, \lambda^{(2)}$ differ are also considered (cf. Section 3).

To the author's knowledge results of this kind are not common, since he is able to cite only a result in van Duyn and Peletier [1], related to a

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similarity solution of Eq. (1.1), and the remarks of Boley's in the survey article [2], based in essence on the assumption $u^{(1)} \geq u^{(2)}$. This author stresses the usefulness of comparison results in which the actual temperature dependence of, say, a melting process could be replaced by a simpler one, thus providing easily obtainable bounds that may help checking the accuracy of an approximate numerical solution [2, pp. 215–216].

Classical solutions of Eq. (1.1) are considered, with the observation that monotone solutions with jump discontinuous data at $t = 0$ are also admitted. A drawback of the method employed is the need to know the boundary values of the solutions $u^{(i)}(x,t)$ on the parabolic boundary $\partial_p G = \partial G \setminus \{(x,T)\}$.

The assumptions made throughout are the following:

$$k^{(i)} \in C^{1+\alpha},\ \alpha > 0;\ 0 < \mu \leq k^{(i)}(z) \text{ for all } z, \tag{1.2}$$

$$u, u_x, u_{xx}, u_t \in C(G), \tag{1.3}$$

$$u, u_x \in C(\overline{G_\tau}), \qquad G_\tau = \{(x,t) \in G\colon 0 < \tau < t\}, \text{ for every } \tau > 0, \tag{1.4}$$

while $k(u)u_x$ is assumed integrable on $\partial_p G$ for $0 < t \leq T$. (That is, the total heat flux entering the spatial domain through the boundary is finite).

These assumptions, together with the monotonicity of $u(x,t)$ allow one to take u, t as independent variables, thus yielding a new equation for $x = x(u,t)$,

$$\left(\frac{k(u)}{x_u(u,t)}\right)_u = -x_t(u,t).$$

Upon integration, this equation gives

$$V_t(u,t) \cdot V_{uu}(u,t) = -k(u), \tag{1.5}$$

for a function $V(u,t)$ related to $\int^u x(v,t)\,dv$. Standard maximum and comparison principles are applicable to Eq. (1.5).

This approach was announced—as a uniqueness result— in the communication [3].

The plan of the work is the following. Section 2 includes the main developments; Section 3, applications of the latter. Qualitative properties of the heat flux $k(u)u_x$ are included in Section 4, together with a revision of the central argument of Section 2, which is cleared of simplifying assumptions. Some final comments on Fokker–Planck equations and on particular solutions of Eq. (1.5) are included in Section 5.

2. The Main Inequality

To detach the main idea of this paper from technical details, the following problem will be considered first. Let $u(x,t)$ be the solution of

$$\begin{aligned} u_t = (k(u)u_x)_x \qquad & \text{in } G = \{(x,t) : x_1 < x < x_2, 0 < t \leq T\}, \\ & \text{(i) } u(x_1,t) = \psi_1(t), \\ & \text{(ii) } u(x_2,t) = \psi_2(t), \\ & \text{(iii) } u(x,0) = \psi_0(x), \end{aligned} \tag{2.1}$$

where $x_1 < x_2$ are constant, $u(x,t)$ is monotone strictly decreasing in the space variable for each $t, 0 < t \leq T$, and therefore $\psi_1(t) > \psi_2(t)$. Clearly it is necessary that $\psi_0(x)$ be also a monotone function of x, and in this section it will be assumed to be strictly so.

For this section's sake it may be assumed that the data ψ is in fact continuous and monotone decreasing as the boundary of the (x,t) domain G is traversed in a direct sense, i.e., ψ_1 decreases with t, ψ_0 is a decreasing function of x, as said, ψ_2 decreases with increasing t. These hypotheses are in fact sufficient for the stated monotonicity of u (cf. Section 4, Theorem 4, Remark 1).

The functions ψ_1, ψ_2 could, in fact, have side limits at zero, $\psi_1(0+) > \psi_0(x_1), \psi_2(0+) < \psi_0(x_2)$, if they are uniformly Lipschitz continuous on $(0,T]$.

The justification of all manipulations done will be included in Section 4.

Let w belong to the range of the solution $u(x,t)$. For every $t \in (0,T]$ put

$$\begin{aligned} & \text{(i) } x = r(t;w) \text{ if } (x,t) \in G \text{ and } u(x,t) = w, \\ & \text{(ii) } r(t;w) = x_1 \text{ if } w \geq \psi_1(t), \\ & \text{(iii) } r(t;w) = x_2 \text{ if } w \leq \psi_2(t). \end{aligned} \tag{2.2}$$

The function thus obtained is continuous in $(0,T]$, and coincides with the level curve $\{(x,t) \in G : u(x,t) = w\}$ whenever $x_1 < r(t:w) < x_2$.

Let $t \in (0,T], u, w$ be given such that $\psi_1(t) > u, w \geq \psi_2(t)$, and assume $u > w$ to be specific. Clearly $r(s;u) \leq r(s;w)$ in $(0,T]$, with strict inequality at t. Integration of Eq. (2.1) in $\{(x,s) \in G : u > u(x,s) > w, 0 < s < t\}$ gives

$$\begin{aligned} \int_0^t (k(u)u_x)(r(s;w),s)ds &- \int_0^t (k(u)u_x)(r(s;u),s)\,ds \\ &= \int_{r(t;u)}^{r(t;w)} u(x,t)dx - \int_{r(0;u)}^{r(0;w)} u(x,0)dx - u(r(0;u) \\ &\quad - r(t,u)) + w(r(0;w) - r(t;w))). \end{aligned} \tag{2.3}$$

Taking u, s as independent variables in this formula gives a new right-hand side for (2.3), namely,

$$\int_{u(r(t;w),t)}^{u} x(v;t)dv - \int_{u(r(0;w),0)}^{u(r(0;u),0)} x(v;0)dv + r(0;w)(w - u(r(0;w),0)) - r(0;u)(u - u(r(0;u),0)) + r(t;w)(u(r(t;w),t) - w). \tag{2.4}$$

It is assumed from now on that (u,t) is an interior point to $\tilde{G} = \{(u,t) : \psi_2(t) < u < \psi_1(t), 0 < t \leq T\}$.

The formula above can therefore be written

$$\int_{u(r(t;w),t)}^{u} x(v,t)dv + \int_0^t (-k(u)u_x)(r(s;w),s)ds + r(t;w)(u(r(t;w),t) - w) = \int_0^t (-k(u)u_x)(r(s;u),s)ds + I(u;w), \tag{2.5}$$

the first term on the right-hand side being a function of u and t, while $I(u,w)$ does not depend on t, and w is thought of as a parameter. Put

$$V(u,t) = \int_{u(r(t;w),t)}^{u} x(v,t)dv + \int_0^t (-k(u)u_x)(r(s;w),s)\,ds + r(t;w)(u(r(t;w),t) - w). \tag{2.6}$$

It follows, using (2.5) and the remark above

$$V_t(u,t) = (-k(u)u_x)(r(t;u),t) = (-k(u)u_x)(u,t) = -\frac{k(u)}{x_u(u,t)}, \tag{2.7}$$

$$\text{i.e., } V_t(u,t) \cdot V_{uu}(u,t) = -k(u), (u,t) \in \tilde{G}. \tag{2.8}$$

Boundary conditions for this parabolic equation in $\tilde{G}$ are, according to (2.1) (i)–(iii),

$$\begin{aligned} &\text{(i) } V(u,0) = \int_{u(r(0;w),0)}^{u} x(v,0)dv + r(0;w)(u(r(0;w),0) - w), \\ &\quad \psi_2(0) < u < \psi_1(0), \\ &\text{(ii) } V_u(\psi_1(t),t) = x_1, \\ &\text{(iii) } V_u(\psi_2(t),t) = x_2, \qquad 0 < t \leq T. \end{aligned} \tag{2.9}$$

The main result of this note is the following. It is assumed that $k^{(1)}(z), k^{(2)}(z)$ are coefficients satisfying (1.2) and such that $k^{(1)}(z) \geq k^{(2)}(z)$ for every z.

For the corresponding solutions $u^{(1)}(x,t), u^{(2)}(x,t)$ of (2.1) (i)–(iii), the respective functions $V^{(1)}(u,t)$, $V^{(2)}(u,t)$ satisfy Eqs. (2.8),

$$V_t^{(i)} V_{uu}^{(i)} = -k^{(i)}(u), \qquad i = 1,2,$$

and the boundary conditions (2.9) (i)–(iii). The parameter w remains arbitrary, but fixed throughout.

Thus $W(u,t) = V^{(1)}(u,t) - V^{(2)}(u,t)$ satisfies

$$(-V_{uu}^{(2)})W_t - (V_t^{(1)})W_{uu} \geq 0, \qquad \text{or} \qquad W_t - a(u,t)W_{uu} \geq 0, \tag{2.10}$$

with boundary conditions

$$\text{(i) } W(u,0) = 0, \qquad \text{(ii) } W_u(\psi_1(t),t) = 0, \qquad \text{(iii) } W_u(\psi_2(t),t) = 0.$$

Here $a(u,t) = V_t^{(1)}(u,t)/(-V_{uu}^{(2)}(u,t)) = k^{(1)}(u) \cdot u_x^{(1)}(x^{(1)},t) \cdot u_x^{(2)}(x^{(2)},t)$, these derivatives evaluated at points where $u^{(1)}(x^{(1)},t) = u^{(2)}(x^{(2)},t) = u$.

Therefore $0 < a(u,t) \leq$ constant by assumption ($u_x^{(i)} \in C(\overline{G})$) and it follows $W(u,t) \geq 0$ (cf. Walter [5, Chap. IV, 31, VI Corollary], for instance. In fact, it is enough to assume $u_x^{(i)} \in C(\overline{G}_\tau)$ for any $\tau > 0$, and employ the boundary point theorem (Protter and Weinberger [6, Chap. 3]), for a (negative) minimum).

That is, $V^{(1)}(u,t) \geq V^{(2)}(u,t)$ in $\tilde{G}$, and returning to definition (2.6) this means

$$\int_{u^{(1)}(r^{(1)}(t;w),t)}^{u} x^{(1)}(v,t)\,dv + \int_0^t \left(-k^{(1)}(u^{(1)})u_x^{(1)}\right)\left(r^{(1)}(s;w),s\right) ds$$
$$\geq \int_{u^{(2)}(r^{(2)}(t;w),t)}^{u} x^{(2)}(v,t)\,dv + \int_0^t \left(-k^{(2)}(u^{(2)})u_x^{(2)}\right)\left(r^{(2)}(s;w),s\right) ds,$$

recalling that $u^{(1)}(r^{(1)}(t;w),t) = u^{(2)}(r^{(2)}(t;w),t)$. Putting now u equal to this value gives the result.

THEOREM 1. *If $k^{(1)}(z) \geq k^{(2)}(z)$ for all z, then*

$$\int_0^t \left(-k^{(1)}(u^{(1)})u_x^{(1)}\right)\left(r^{(1)}(s;w),s\right) ds \geq \int_0^t \left(-k^{(2)}(u^{(2)})u_x^{(2)}\right)\left(r^{(2)}(s;w),s\right) ds. \tag{2.11}$$

Recall that $r^{(i)}(s; w)$ coincides with the level curve $\{(x, t) \in G : u^{(i)}(x,t) = w\}$ whenever $(r^{(i)}(s; w), s) \in G$. Thus (2.11) estimates the total flux of heat across these curves in the lapse $(0, t]$. If the level curves lie wholly in G, (2.11) reads

$$\int_0^t \left(-k^{(1)}(w)u_x^{(1)}\right)\Big|_{u^{(1)}=w} ds \geq \int_0^t \left(-k^{(2)}(w)u_x^{(2)}\right)\Big|_{u^{(2)}=w} ds. \tag{2.12}$$

For the case $u^{(i)} = u^{(i)}(x/t^{1/2}), 0 < x < \infty, 0 < t \leq T,$

$$u^{(i)} = b > 0 \text{ at } x = 0, \qquad u^{(i)} \to 0 \text{ as } x \to \infty, \qquad u^{(i)} = 0 \text{ at } t = 0$$

(i.e., for similarity solutions of Eq. (2.1) in the strip), a result equivalent to (2.12) was obtained by van Duyn and Peletier as an isolated theorem in [1]. These authors phrased (2.12) as follows:

$$\int_0^u \sigma_1(s) ds > \int_0^u \sigma_2(s)\, ds,$$

where σ_i is the function inverse to $u^{(i)}(\eta)$. Their proof makes use of the fact that $u^{(i)}(\eta)$ are functions of a real variable. In this paper's context two difficulties are apparent: the discontinuity of the data at (0, 0), and the infinite spatial domain $(0, \infty)$. The first is taken care of as noted (cf. Section 4, also); for the second it should be observed that $k(u^{(i)})(du^{(i)}/d\eta) \to 0$ as $\eta \to \infty$ and the level curves are parabolae $x = \eta \cdot t^{1/2}$. It is seen that $V^{(i)}(u, t) = \int_0^u x^{(i)}(v, t) dv, 0 < u < b$ (cf. also Section 5).

3. Applications

Inequality (2.11) can be put to use in some moving boundary problems when, say, $x_2 = x_2(t)$ is in principle an unknown portion of the parabolic boundary—except for the initial position $x_2(0)$—but an interphase condition

$$- k(u)u_x + f(t) = \lambda \dot{x}_2(t), \qquad \lambda > 0$$

takes place whenever $u = u_0 =$ constant. This is the case of the classical one-phase Stefan problem (cf. [4]), formulated as follows.

To find $u(x, t)$ and $x_2(t)$, u satisfying (1.3), (1.4) such that $u_t = (k(u)u_x)_x$ in G, $k \in C^{1+\alpha}$ (cf. (1.2)) and $u(x_1, t) = \psi_1(t) > u_0, u(x, 0) = \psi_0(x), x_1 \leq x \leq x_2(0), x_2(0)$ given, $u(x_2(t), t) = u_0$, and

$$(-k(u)u_x)(x_2(t), t) + f(t) = \lambda \dot{x}_2(t) \tag{3.1}$$

Assuming $u(x,t)$, monotone decreasing in x, and $x_2(t)$ is a solution of this problem, then the function (cf. (2.6), with $w = u_0$)

$$V(u,t) = \int_{u_0}^{u} x(v,t)\,dv + \int_0^t (-k(u)u_x)(x_2(s),s)\,ds + \int_0^t f(s)\,ds + \lambda x_2(0)$$

$$= \int_{u_0}^{u} x(v,t)\,dv + \lambda x_2(t) \tag{3.2}$$

satisfies (cf. (2.5), (2.8))

$$V_t = -\frac{k(u)}{V_{uu}} + f(t) \tag{3.3}$$

with the boundary conditions

(i) $V(u,0) = \int_{u_0}^{u} \psi_0^{-1}(v)\,dv + \lambda x_2(0)$,

(ii) $V_u(\psi_1(t),t) = x_1$,

(iii) $V(u_0,t) = \lambda V_u(u_0,t)$.

This is a boundary value problem of mixed type for Eq. (3.3) in the domain $\tilde{G} = \{(u,t): u_0 < u < \psi_1(t), 0 < t \le T\}$.

THEOREM 2. *Let $u^{(1)}(x,t), u^{(2)}(x,t)$ be solutions decreasing in the space variable of two Stefan problems as stated, whose respective coefficients and boundary conditions satisfy*

$k^{(1)}(z) \ge k^{(2)}(z)$ *for all* z, $\quad \psi_1^{(1)}(t) \ge \psi_1^{(2)}(t) > u_0 = u^{(i)}(x_2^{(i)}(t),t)$, $i = 1,2$,

$\psi_0^{(1)}(x) \ge \psi_0^{(2)}(x), x_2^{(1)}(0) \ge x_2^{(2)}(0), f^{(1)}(t) \ge f^{(2)}(t)$, *$\lambda$ being the same in both problems. Then* $x_2^{(1)}(t) \ge x_2^{(2)}(t)$ *for all* t.

Proof. It is straightforward to verify that the function $W(u,t) = V^{(1)}(u,t) - V^{(2)}(u,t)$ is a solution of inequality (2.10) in the domain $\{(u,t): u_0 < u < \psi_1^{(2)}(t), 0 < t \le T\}$, subjected to the boundary conditions

(i) $W(u,0) = \int_{u_0}^{u} (\psi_0^{(1)-1} - \psi_0^{(2)-1})\,dv + \lambda(x_2^{(1)}(0) - x_2^{(2)}(0)) \ge 0$,

(ii) $W_u(\psi_1^{(2)}(t),t) = x^{(1)}(\psi_1^{(2)}(t),t) - x_1 \ge 0$,

(iii) $W(u_0,t) = \lambda W_u(u_0,t)$ (cf. 3.3)).

These conditions imply $W(u,t) \ge 0$ in $u_0 < u < \psi_1^{(2)}(t)$. In fact W cannot take a negative minimum on $u = \psi_1^{(2)}(t)$, $t > 0$, for $W_u < 0$ there, nor it can take a negative minimum on $u = u_0$, $t > 0$, because W_u would then be < 0

at that point. It follows by continuity that

$$0 \le W(u_0,t) = V^{(1)}(u_0,t) - V^{(2)}(u_0,t) = \lambda\big(x_2^{(1)}(t) - x_2^{(2)}(t)\big),$$

whence the theorem.

If $u^{(1)}, u^{(2)}$ are solutions as considered in the theorem, with same u_0 but $\lambda^{(1)} \ge \lambda^{(2)}$, the difference $W = V^{(1)} - V^{(2)}$ satisfies (2.10) with boundary conditions

(i) $W(u,0) = \int_{u_0}^{u} (\psi_0^{(1)-1} - \psi_0^{(2)-1})dv + \lambda^{(1)}x_2^{(1)}(0) - \lambda^{(2)}x_2^{(2)}(0) \ge 0,$

(ii) $W_u(\psi_1^{(2)}(t),t) = x^{(1)}(\psi_1^{(2)}(t),t) - x_1 \ge 0,$

(iii) $W(u_0,t) = \lambda^{(1)}x_2^{(1)}(t) - \lambda^{(2)}x_2^{(2)}(t) = \lambda^{(1)}(x_2^{(1)}(t) - x_2^{(2)}(t)) + (\lambda^{(1)} - \lambda^{(2)})x_2^{(2)}(t) = \lambda^{(1)}W_u(u_0,t) + (\lambda^{(1)} - \lambda^{(2)})x_2^{(2)}(t).$

The previous reasoning shows $W(u,t) \ge 0$, because $W(u_0,t) \ge \lambda^{(1)}W(u_0,t)$. Thus, for $u = u_0$,

$$\lambda^{(1)}x_2^{(1)}(t) \ge \lambda^{(2)}x_2^{(2)}(t), \qquad \text{or} \qquad x_2^{(1)}(t) \ge \frac{\lambda^{(2)}}{\lambda^{(1)}}x_2^{(2)}(t).$$

This lower bound for the (supposedly unknown) boundary $x_2^{(1)}(t)$ is—admittedly—insufficient and may even be irrelevant if, e.g., $(\lambda^{(2)}/\lambda^{(1)})x_2^{(2)}(t) < x_1$. To improve this bound the following observation is in order: the possibility of comparing $x_2^{(1)}(t)$ and $x_2^{(2)}(t)$ will depend on the sign of $\int_0^t(-k(u)u_x)\big(x_2^{(2)}(s),s\big)\,ds + \int_0^t f^{(2)}(s)ds$, that is, on whether or not $x_2^{(2)}(t) \ge x_2^{(2)}(0)$. This sign is physically related to a supply or removal of heat at the interphase—represented by the additional term $f(t)$—which is not due to conduction. In the sequel it will be assumed that $\lambda^{(1)} \ge \lambda^{(2)}, x_2^{(1)}(0) = x_2^{(2)}(0) = x_2(0), x_2^{(2)}(t) \ge x_2^{(2)}(0)$ for $t > 0$.

Put

$$V^{(i)}(u,t) = \int_{u_0}^{u} x^{(i)}(v,t)dv + \int_0^t\big(-k^{(i)}(u^{(i)})u_x^{(i)}\big)\big(x_2^{(i)}(s),s\big)ds + \int_0^t f^{(i)}(s)ds.$$

Then $W = V^{(1)} - V^{(2)}$ satisfies (2.10) with boundary conditions

(i) $W(u,0) = \int_{u_0}^{u} (\psi_0^{(1)-1} - \psi_0^{(2)-1})dv \ge 0,$

(ii) $W_u(\psi_1^{(2)}(t),t) \ge 0,$

(iii)
$$\begin{aligned} W(u_0,t) &= \lambda^{(1)}(x_2^{(1)}(t) - x_2^{(2)}(0)) - \lambda^{(2)}(x_2^{(2)}(t) - x_2^{(2)}(0)) \\ &= \lambda^{(1)}(x_2^{(1)}(t) - x_2^{(2)}(t)) + (\lambda^{(1)} - \lambda^{(2)})x_2^{(2)}(t) \\ &\quad - (\lambda^{(1)} - \lambda^{(2)})x_2(0) \\ &\ge \lambda^{(1)}W_u(u_0,t). \end{aligned}$$

Again $W(u,t) \ge 0$, and from (iii) it follows

COROLLARY. *If in addition to the hypotheses of the theorem it is assumed that* $x_2^{(2)}(t) \geq x_2^{(2)}(0), x_2^{(1)}(0) = x_2^{(2)}(0) = x_2(0), \lambda^{(1)} \geq \lambda^{(2)}$, *then*

$$\left(1 - \frac{\lambda^{(2)}}{\lambda^{(1)}}\right) x_2(0) + \frac{\lambda^{(2)}}{\lambda^{(1)}} x_2^{(2)}(t) \leq x_2^{(1)}(t).$$

If instead $\lambda^{(1)} \leq \lambda^{(2)}$, *and* $x_2^{(2)}(t) \leq x_2(0)$ *for* $t > 0$,

$$x_2^{(2)}(t) \leq \frac{\lambda^{(1)}}{\lambda^{(2)}} x_2^{(1)}(t) + \left(1 - \frac{\lambda^{(1)}}{\lambda^{(2)}}\right) x_2(0).$$

So far, the examples have dealt with interphase conditions at the lowest temperature. An interphase condition at the highest temperature (cf. Boley [2, p. 216])

$$\begin{aligned} &- k(u_m) u_x = f(t) - \lambda \dot{x}(t), \\ &u(x(t), t) = u_m \geq u \text{ in } G, \qquad x(0) = x_1, \end{aligned} \tag{3.4}$$

is related to problems of solidification $(x(t) \leq x_1)$ and ablation $(x(t) \geq x_1)$. A simple one is

$$\begin{aligned} &u_t = (k(u)u_x)_x \text{ in } x(t) < x < x_2, \quad x(0) = x_1, 0 < t \leq T, \\ &u(x,0) = \psi_0(x), \qquad u(x_2, t) = 0, \quad -(k(u)u_x)(x(t),t) = f(t) - \lambda \dot{x}(t), \\ &u(x(t),t) = u_m. \end{aligned}$$

Two such problems will be considered, with unknowns, coefficients and data $k^{(i)}, u^{(i)}(x,t), x^{(i)}(t), \psi_0^{(i)}, \lambda^{(i)}, f^{(i)}(t), i = 1,2,$ where it is assumed $x^{(1)}(0) = x^{(2)}(0) = x_1$. Let $x^{(i)}(u,t)$ denote the corresponding inverse functions. An easy reckoning gives

$$\begin{aligned} V^{(i)}(u,t) &\equiv \int_{u_m}^{u} x^{(i)}(v,t)\, dv + \int_0^t \left(-k(u^{(i)})u_x^{(i)}\right)\left(r^{(i)}(s; u_m), s\right) ds \\ &= \int_0^t \left(-k(u^{(i)})u_x^{(i)}\right)\left(r^{(i)}(s;u),s\right) ds + \int_{u_m}^{u} \psi_0^{(i)^{-1}}(v)\, dv. \end{aligned}$$

$V^{(i)}$ satisfy Eq. (2.8). Assuming throughout $k^{(1)} \geq k^{(2)}$, $\psi_0^{(1)} \leq \psi_0^{(2)}$, $W = V^{(1)} - V^{(2)}$ is a solution of (2.10) in $\tilde{G} = \{(u,t): 0 < u < u_m, 0 < t \leq T\}$, with boundary conditions $W(u,0) = \int_{u_m}^{u} (\psi_0^{(1)^{-1}} - \psi_0^{(2)^{-1}}) dv \geq 0$ $(0 \leq u \leq u_m)$, $W_u(0,t) = 0$, and

$$\begin{aligned} W(u_m, t) = -\lambda^{(1)} W_u(u_m, t) &+ (\lambda^{(1)} - \lambda^{(2)})(x_1 - x^{(2)}(t)) \\ &+ \int_0^t (f^{(1)}(s) - f^{(2)}(s))\, ds. \end{aligned}$$

The following theorem holds.

THEOREM 3. *Let* $k^{(1)}(z) \geq k^{(2)}(z), \psi_0^{(1)}(x) \leq \psi_0^{(2)}(x)$,

$$\int_0^t (f^{(1)}(s) - f^{(2)}(s))ds \geq 0, \qquad 0 < t \leq T.$$

Then (1) *if* $\lambda^{(1)} \geq \lambda^{(2)}, x^{(2)}(t) \leq x_1$,

$$x^{(1)}(t) \leq \frac{\lambda^{(2)}}{\lambda^{(1)}} x^{(2)}(t) + \left(1 - \frac{\lambda^{(2)}}{\lambda^{(1)}}\right)x_1 + \frac{1}{\lambda^{(1)}}\int_0^t (f^{(1)}(s) - f^{(2)}(s)\, ds;$$

and (2) *if* $\lambda^{(1)} \leq \lambda^{(2)}, x^{(2)}(t) \geq x_1$,

$$\frac{\lambda^{(1)}}{\lambda^{(2)}} x^{(1)}(t) + \left(1 - \frac{\lambda^{(1)}}{\lambda^{(2)}}\right)x_1 \leq x^{(2)}(t) + \frac{1}{\lambda^{(2)}}\int_0^t (f^{(1)}(s) - f^{(2)}(s))\, ds.$$

Remark. Both inequalities above stand without the integrals on the right-hand sides if it is assumed that $f^{(1)}(s) \leq f^{(2)}(s)$ for all s. The proof is accomplished by including the term $-\int_0^t f^{(i)}(s)ds$ in the definition of $V^{(i)}$ (as done in the demonstration of the corollary to Theorem 2. A corresponding remark applies to that corollary).

4. ANALYSIS

This section is devoted to the justification of most of the assumptions made in Sections 2, 3.

THEOREM 4. *Let* $0 < \mu \leq k(z) \in C^{1+\alpha}$, $\alpha > 0$, *and let* $u(x,t)$ *be a solution of* $u_t = (k(u)u_x)_x$. *Then* $F(x,t) = (k(u)u_x)(x,t)$ *satisfies*

$$F_t(x,t) = (k(u(x,t))F_x(x,t))_x. \tag{4.1}$$

That is, the heat flux $(-F)$ "diffuses" with diffusivity $k(u(x,t))$.

Proof. Let $U(z) = \int^z k(v)dv$ be a primitive of $k(z)$. Clearly $u_t = (U(u))_{xx}$, whence

$$(U(u))_t = k(u)u_t = k(u)(U(u))_{xx}, \tag{4.2}$$

and therefore $U(u)$ is a solution of a parabolic equation with $C^{1+\alpha}$ coefficients. It is known (cf. Friedman [7, Chap. 3, Sect. 5]) that $(U(u))_{xxx}$, $(U(u))_{xt} \in C^\alpha$, and from (4.2) follow $(U(u))_{tx} \in C^\alpha$, $(U(u))_{xt} = (U(u)_{tx}$, and finally Eq. (4.1).

Remarks. (1) Results on the shape of profiles $u = u(x,t)$, $t =$ constant are known provided the data is of the first kind and u is continuous in $\overline{G}$ (cf. Walter [5, Chap. IV, Sect. 27]; Redheffer and Walter [8]). In particular, for u to be monotone decreasing in x it is sufficient to assume that the boundary values of the solution $u(x,t)$ of (2.1) in G are decreasing as $\partial_p G$ is traversed in direct sense.

(2) The results just cited are based on the maximum or comparison principles for the respective parabolic operator. Together with Theorem 4 they may be used to gather information on the behavior of solutions to some mixed boundary value problems for Eq. (2.1). If $u_x(x_1,t), u_x(x_2,t), u(x,0)$ are prescribed on $\partial_p G$, the changes of sign of $u_x(x,t)$ for fixed t are estimated by those of $k(u)u_x$ on $\partial_p G$—computed from $u(x,0)$ on $t = 0$ (cf. Walter [5]). In particular, $u_x(x_1,t) \leq 0, u_x(x_2,t) \leq 0, u(x,0)$ monotone decreasing imply $u(x,t)$ is monotone decreasing in x for all time. If $F = k(u)u_x$ is a datum on $\partial_p G$, increasing as $\partial_p G$ is traversed counterclockwise, it will be so in G for fixed t, whence $u_t = F_x \geq 0$ in G.

Assume $u_x(x_2,t) = 0, k(u)u_x = \phi(u)$ on $x = x_1$, with ϕ a monotone function of u that vanishes at $u = U$. Then if $u(x,0) < U$ for all x, u remains $< U$ for all t. In fact, $u = U$ on x_1 first at $t = t_1$ implies $u_x(x_1,t_1) = 0$ at a maximum of the solution u, contradicting the boundary point theorem [6, Chap. 3]. Thus $F = k(u)u_x$ has a constant sign on $x = x_1, x = x_2$. If $u_x(x,0)$ is of that sign and $u(x,0) < U$, u is a monotone function of x for every t.

(3) It should be mentioned here that comparison results are available for a class of parabolic operators including

$$u_t = (k(x,t,u)u_x)_x,\ u_t = (k(x,t,u)u_x)_x + (h(x,t,u))_x,$$

within a class of generalized solutions admitting discontinuous boundary values, provided integrability assumptions are made on u_x, u_t.

Returning to the monotone decreasing solution $u(x,t)$ of (2.1), it is clear that $u_x(x,t) \leq 0$.

Furthermore,

THEOREM 5. *Either $u_x(x,t) < 0$ in G, or there is a $t_0 \in (0,T]$ such that $u =$ constant for $0 < t \leq t_0$.*

Proof. $F(x,t) = (k(u)u_x)(x,t) \leq 0$ in G satisfies Eq. (4.1). This equation has a strong maximum principle (cf. [5–7]) and the conclusion follows thereby.

Remark. It has been assumed that $0 < \mu \leq k(u)$. For certain applications it may be convenient to suppose $k(u) > 0$ if $u > 0, k(0) = 0$. In this case both Theorems 4 and 5 are valid wherever $u > 0$.

It follows from Theorem 2 that the level curves $\{(x,t) : u(x,t) = C\}$ of a solution to Eq. (2.1) are smooth curves $x = x(t)$ provided $(x(t),t) \in G$. By the maximum principle it follows that, if the curve is not void, the value C must be attained also on $\partial_p G$, possibly in more than one point. It this general setting, the possibility that $x(t)$ oscillated, with $\liminf x(t) <$

$\limsup x(t)$, as $t \to 0^+$—and only in this case—must be admitted; it must follow then that $u(x,0) = C$ in $[\liminf x(t), \limsup x(t)]$. It is also clear that $u(x_1,t) > u(x_2,t)$ for every $t \in (0,T]$ unless u = constant up to certain $t_0 > 0$.

Let now $u(x,t)$ be a solution of (2.1) in $G = \{(x,t): x_1 < x < x_2, 0 < t \le T\}$ (the moving boundary cases in Section 3 require only minor alterations of the following arguments). It will be assumed that the values $\psi_0(x), \psi_1(t), \psi_2(t)$, taken by $u(x,t)$ on $t = 0, x = x_1, x = x_2$, are uniformly Lipschitz continuous functions, $\psi_0(x)$ decreasing, with $\psi_1(0^+) \ge \psi_0(x_1) \ge \psi_0(x_2) \ge \psi_2(0^+)$. No assumption on the monotonicity of ψ_1 or ψ_2 is made.

The functions $r(t;w)$ defined in (2.2), for fixed w, are smooth functions of t if $(x(w,t),t) \in G$, while continuous for every t by definition.

The main arguments in Section 2 rely on formulae (2.3), (2.4), whose proof—in the general case—now follows. Let $t \in (0,T]$ be fixed, $\psi_1(t) > u > w \ge \psi_2(t)$: then $x_1 < r(t;u) < r(t;w) \le x_2$. Integration of Eq. (2.1) in $G_{uw\delta} = \{(x,s) \in G : u > u(x,t) > w, 0 < \delta < s < t\}$ easily give the left-hand side of formulae (2.3), (2.4), with 0 replaced with $\delta > 0$. For the evaluation of the integral $\iint_{G_{uw\delta}} u_t\,dx\,dt$ the following result is needed.

LEMMA 1. *Assuming* $x_1 \le a \le x_2$; $a < r(s;w), c < s < d, a = r(c;w) = r(d;w)$, $H = \{(x,s): a < x < r(s;w), c < s < d\} \subset G$ *then* $\iint_H u_t\,dx\,dt = 0$ (*and corresponding result when* $a > x > r(s;w)$).

Proof. For every x, $H \cap \{(x,s): s \in (0,T)\}$ is an at most denumerable (in fact, finite a.e. x) union of open segments at the end-points s_i, s^i of which $u = w$ = constant. Therefore,

$$\int \chi_H(x,s) u_t(x,s)\,ds = \sum \int_{s_i}^{s^i} u_t\,ds = \sum \left(u(x,s^i) - u(x,s_i)\right) = 0$$

for every x, χ_H denoting the characteristic function.

Remark. This lemma obviously implies

$$\int_c^d (k(u)u_x)(r(s;w),s)\,ds = \int_c^d (k(u)u_x)(a,s)\,ds.$$

LEMMA 2.

$$\iint_{G_{uw\delta}} u_t\,dx\,dt = \int_{r(t;u)}^{r(t;w)} u(x,t)\,dx - \int_{r(\delta;u)}^{r(\delta;w)} u(x,\delta)\,dx$$
$$- u(r(\delta;w) - r(t;u)) + w(r(\delta;w) - r(t;w)). \qquad (4.3)$$

Proof. By application of the lemma, and without modifying the left-hand side above, one can adjoin to $G_{uw\delta}$ and remove from it sets of the form

$$H = \{(x,s) : a < x < r(s;u), c < s < d\} \text{ and}$$
$$H = \{(x,s) : a > x > r(s;u), c < s < d\},$$
$$H = \{(x,s), a < x < r(s;w), c < s < d\} \text{ and}$$
$$H = \{(x;s) : a > x > r(s;w), c < s < d\}.$$

This can be done in such a way as to transform $G_{uw\delta}$ into an open set bounded, on the left by a monotone curve composed of vertical segments and monotone portions of the curve $r(s;u)$, and joining $(r(t;u),t)$ to $(r(\delta;u),\delta)$, and on the right by a similar monotone curve joining $(r(t;w),t)$ and $(r(\delta;w),\delta)$. Integration on this new domain yields the right-hand side.

It is clear that the change of independent variables $(x,t) \to (u,t)$ maps the domain G onto a domain $\tilde{G}$ in the (u,t)-plane, whose parabolic boundary is composed of the curves $u = \psi_1(t), u = \psi_2(t)$ for $0 < t \leq T, t = 0$ for $\psi_2(0+) \leq u \leq \psi_1(0+)$. Introducing these new variables in (4.3) gives

$$\iint_{G_{uw\delta}} u_t \, dx \, dt = \int_{u(r(t;w),t)}^{u} x(v,t) dv + r(t,w)(u(r(t;w),t) - w) + I_\delta(u;w). \tag{4.4}$$

Here

$$I_\delta(u,w) = -\int_{r(\delta;u)}^{r(\delta;w)} u(x,\delta) dx - u \cdot r(\delta;u) + w \cdot r(\delta;w).$$

In order to let $\delta \to 0+$, it should be observed first that the integrals $\int_0^t (k(u)u_x)(r(s;u),s)\, ds, \int_0^t (k(u)u_x)(r(s;w),s)\, ds$ on the left-hand side of (2.3) exist due to assumption (1.4) and the remark following Lemma 1. On the other hand, taking into account the possible oscillation of $r(\delta;w)$ or $r(\delta;u)$ as $\delta \to 0+$, the term $I_\delta(u;w)$ can be written as follows, where $b = \limsup_{\delta \to 0^+} r(\delta;u) \leq c = \liminf_{\delta \to 0^+} r(\delta;w)$, and $w < u$ for simplicity

$$I_\delta(u;w) = -\int_{r(\delta;u)}^{b} (u(x,\delta) - u) dx - \int_b^c u(x,\delta)\, dx - \int_c^{r(\delta;w)} (u(x,\delta) - w) dx - ub + wc.$$

It is now easily seen that the oscillating terms actually have limit zero as $\delta \to 0+$, thus yielding—as before—an expression $I(u,w)$ independent of t in which w is to be thought of as a parameter. The second term on the right-hand side of (4.4) is zero if $x_1 < r(t;w) < x_2$, and $x_i(\psi_i(t) - w)$ if $r(t;w) = x_i$. If the strict inequalities hold for certain $t \to 0+$, this term has limit zero.

Observe finally that the boundary point theorem for a minimum is valid for (2.10) if the boundary is uniformly Lipschitz continuous.

5. Final Comments

Fokker–Planck's equation of the theory of infiltration can be studied in a similar manner. The equation is

$$u_t = (k(u)u_x)_x - (h(u))_x$$

and the remarks on maximum and comparison principles, made for the diffusion equation, apply to this one as regards the monotonicity of the solution $u(x, t)$. Furthermore, it is not difficult to see that $F(x,t) = (k(u)u_x)(x,t)$ satisfies

$$F_t = (k(u)\cdot F_x)_x - (h'(u)\cdot F)_x,$$

if sufficient differentiability of k and h is assumed.

If the function h is assumed to be convex, this equation is

$$F_t = (k(u)F_x)_x - h'(u)\cdot F_x + (-h''(u)\cdot u_x)\cdot F,$$

where $(-h''(u)\cdot u_x) \geq 0$. As $F \leq 0$, it is clear that

$$F_t \leq (k(u)F_x)_x - h'(u)\cdot F_x.$$

Therefore (cf. [6, chap. 3]) a maximum $F = 0$ cannot be attained at $(x_0, t_0) \in G$ without having $F = 0$ for $t \leq t_0$ in G, wherever $u > 0$.

The analog of (2.5) for this equation is

$$\begin{aligned} V(u,t) &\equiv \int_{u(r(t;w),t)}^{u} x(v,t)\,dv + \int_0^t(-k(u)u_x)(r(s;w),s)\,ds \\ &\quad - \int_0^t h(u(r(s;w),s))\,ds \\ &= \int_0^t(-k(u)u_x)(r(s;u),s)\,ds + \int_0^t h(u(r(s;u),s))\,ds + I(u,w), \end{aligned}$$

whence the equation, analogous to (2.8), $V_t = h(u) - k(u)/V_{uu}$.

The differential inequality (2.10) is now easily derived, assuming the existence of $V^{(1)}, V^{(2)}$, solutions of (2.8) corresponding respectively to $k^{(1)} \geq k^{(2)}, h^{(1)} \geq h^{(2)}$.

No attempt was made at solving Eq. (2.8) with boundary conditions (2.9), although its nonlinearity is reduced to a product of the derivatives. Particular solutions—for corresponding particular data— can be obtained via separation of variables, e.g.,

(1) Solutions of the form $V = U(u) + u\cdot T(t)$ lead to $x(u,t) = V_u(u,t) = -(1/a)\int^u (k(z)/z)\,dz + at +$ constant, whence the traveling waveshape $u = u(x - at)$;

(2) Put $V = U(u) \cdot T(t)$. It follows $T(t) = (t + a)^{1/2}$, $U \cdot U'' = -2k(u)$. The latter is known to be the equation for $U(u) = \int_0^u \sigma(s) ds$, where σ is the function inverse to $u = u(\eta)$ (cf. van Duyn and Peletier [1], and references therein). Thus $x = U'(u) \cdot (t + a)^{1/2}$ or $x/(t + a)^{1/2} = \eta = \sigma(u)$, yielding the classical similarity solution of Eq. (2.1).

As a particular case of the above, equation $U \cdot U'' = u$ was studied—independently of its relation to diffusion–conduction problems—by Levi and Massera [9] back in 1947.

References

1. C. J. van Duyn and L. A. Peletier, A class of similarity solutions of the nonlinear diffusion equation, *Nonlinear Anal. Theory Methods Appl.* **1**, No. 3 (1977), 223–233.
2. B. A. Boley, An applied overview of moving boundary problems, in "Moving Boundary Problems" (D. G. Wilson, A. D. Solomon, and P. T. Boggs, Eds.), Academic Press, New York, 1978.
3. J. E. Bouillet, Un caso de unicidad para ciertos tipos de difusión no lineal, Communication to the XXV Reunión Anual de la U.M.A., August 1975, *Rev. Un. Mat. Argentina* **27**, No. 4 (1976), 264.
4. L. I. Rubinstein, "The Stefan Problem," AMS Translations, Amer. Math. Soc., Providence, R.I., 1971.
5. W. Walter, "Differential and Integral Inequalities," Springer-Verlag, Berlin, 1970.
6. M. H. Protter and H. F. Weinberger, "Maximun Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, N.J. 1967.
7. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, N. J., 1964.
8. R. M. Redheffer and W. Walter, The total variation of solutions of parabolic differential equations and a maximun principle in unbounded domains, *Math. Ann.* **209** (1974), 57–67.
9. B. Levi and J. L. Massera, Estudio en grande de una ecuación diferencial del segundo orden, *Math. Notae* **7**, 2–3, (1947), 91–155.